# A polynomial-time dynamic programming algorithm for an optimal picking problem in automated warehouses

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Discrete Optimization

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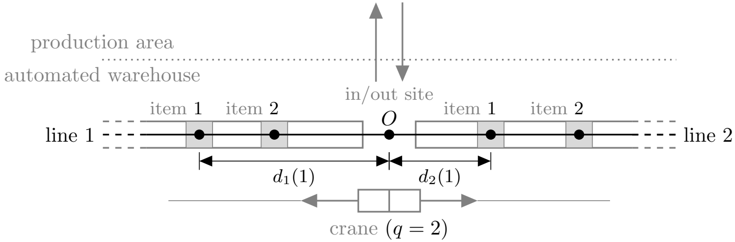
# The warehouse

An Automated Storage/Retrieval System (AS/RS):

- a set of identical storage locations along a single aisle;
- an *origin* *O* at a given position, dividing the rail in two *lines*;
- a crane moving on the rail.



# The order picking problem ($q/1/P/V$)

**Assumption:** Each item has two locations, one on each line.

**Data:**

- a set $N = \{1, 2, \ldots, n\}$ of *items* to be picked-up;
- the distance $d_\ell(i)$ of each item $i \in N$ from $O$ on each line $\ell \in \{1, 2\}$;
- the capacity $q$ of the crane.

**Constraints:** starting from $O$, pick-up all requested items from either line and carry them to $O$, complying with the capacity constraint of the crane in each trip.

**Objective**: minimize the total distance travelled.

# Variables: trips

## Definition (Trips)

A *trip* $T$ is a subset of $N$ of cardinality at most $q$.

The *cost* of a trip $T$ on line $\ell \in \{1, 2\}$ is $C_\ell(T) = \max_{i \in T}\{d_\ell(i)\}$, *i.e.*, half the distance travelled by the crane.

Find a pair $(\mathcal{T}^1, \mathcal{T}^2)$ of sets of non-empty trips such that

- the trips in $\mathcal{T}^1 \cup \mathcal{T}^2$ partition $N$
- the total cost $C(\mathcal{T}) = \sum_{\ell=1}^{2} \sum_{T \in \mathcal{T}^\ell} C_\ell(T)$ is minimum.

# Leading item

## Definition (Leading item)

A *leading item* of a trip $T$ on line $\ell$ is an item in $T$ that is farthest from $O$, *i.e.*, an item $j \in T$ such that $d_\ell(j) = \max_{i \in T}\{d_\ell(i)\}$.

A leading item is not necessarily unique.

$C_\ell(T) = d_\ell(j)$, where $j$ is a leading item of $T$.

# Line assignment

Determining a feasible solution to $q/1/P/V$ consists of

- deciding a *line assignment*, *i.e.*, determining the line where each item must be picked-up;
- grouping the items assigned to the same line into trips.

The clustering problem on each line is solvable to optimality by a greedy algorithm (see [**?**]) with complexity $O(n \log n)$:

- sort the items on each line according to their distance from $O$;
- group them in clusters of $q$, starting from the farthest ones.

Hence, an implicit complete enumeration of feasible solutions can obtained through an implicit complete enumeration of line assignments.

# Compact and complete solutions

For any given line $\ell \in \{1, 2\}$ and any given set $N_\ell$ of items assigned to it, let $\mathcal{T}^\ell = \{T_1^\ell, T_2^\ell, \ldots, T_m^\ell\}$ a set of trips partitioning $N_\ell$.

## Definition (Compact trips and solutions)

A set $\mathcal{T}^\ell$ of trips on line $\ell \in \{1, 2\}$ is *compact* if and only if, for any two distinct trips $T_1, T_2 \in \mathcal{T}^\ell$, either $d_\ell(i) \geq d_\ell(j)\ \forall i \in T_1, \forall j \in T_2$ or $d_\ell(i) \leq d_\ell(j)\ \forall i \in T_1, \forall j \in T_2$.
A feasible solution $(\mathcal{T}^1, \mathcal{T}^2)$ to $q/1/P/V$ is *compact* if and only if $\mathcal{T}^\ell$ is compact for any line $\ell \in \{1, 2\}$.

## Definition (Complete trips and solutions)

Consider a set $\mathcal{T}^\ell$ of $m_\ell$ trips on line $\ell \in \{1, 2\}$ ordered by non-increasing distance of their leading items. $\mathcal{T}^\ell$ is *complete* if and only if its farthest $m_\ell - 1$ trips are made of $q$ items each.
A feasible solution $(\mathcal{T}^1, \mathcal{T}^2)$ to $q/1/P/V$ is *complete* if and only if $\mathcal{T}^\ell$ is complete for $\ell = 1, 2$.

# Compact and complete solutions

The converse is also true.

## Property (Optimality)

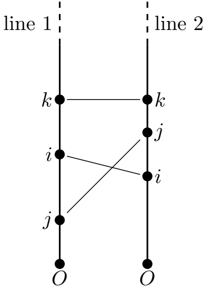
*For any given line assignment $\mathcal{A}$, any solution that is compact and complete is optimal for $q/1/P/F(\mathcal{A})$.*

This property allows to compare *partial* line assignments, to early discard some that cannot correspond to optimal solutions.

# Line assignments and edges

We associate each item in $N$ with an edge.

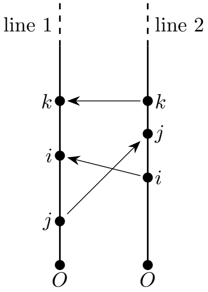
Assigning an item to a line corresponds to orienting its edge.



## Definition (Orientations)

Edge $i \in N$ is *horizontal* if and only if $d_1(i) = d_2(i)$.
A horizontal edge is $\ell$*-oriented* if and only if item $i$ is assigned to line $\ell$.

A non-horizontal edge $i \in N$ is *upward-oriented* (*downward-oriented*) if and only if item $i$ is assigned to the line where it is farther from (closer to) the origin.



# Dominance between line assignments

Consider a line $\ell \in \{1, 2\}$ and a compact and complete set $\mathcal{T}^\ell$ of trips on that line; let $C$ be its cost.

## Property (Replacement)

*If an item $i \in \mathcal{T}^\ell$ is replaced by an item $j \notin \mathcal{T}^\ell$ with $d_\ell(j) \leq d_\ell(i)$, then the cost of a compact and complete set of trips collecting the items in $T^\ell \setminus \{i\} \cup \{j\}$ on line $\ell$ is not larger than $C$.*

Consider a solution with a compact and complete set $\mathcal{T}^\ell$ of trips on each line $\ell \in \{1, 2\}$ and let $C$ be its cost.

## Property (Swap)

*If two items $i \in \mathcal{T}^1$ and $j \in \mathcal{T}^2$ with $d_1(j) \leq d_1(i)$ and $d_2(i) \leq d_2(j)$ are swapped, then the cost of the optimal solution corresponding to the new assignment is not larger than $C$.*

# Dominance between line assignments

## Definition (Dominance)

Given two line assignments $\mathcal{A}$ and $\mathcal{A}'$, $\mathcal{A}$ *dominates* $\mathcal{A}'$ if and only if $\mathcal{A}$ is obtained from $\mathcal{A}'$ through a sequence of the edge reversal operations described in Property 3.

## Property

*The dominance relation of Def. 6 is asymmetric and transitive.*

## Property

*Every instance of $q/1/P/V$ admits at least one optimal solution whose corresponding line assignment is non-dominated.*

This allows to restrict the search for an optimal solution by considering only non-dominated line assignments.

## Definition (Intersecting edges)

Two distinct edges $i \in N$ and $j \in N$ *intersect* if and only if $d_{\ell'}(i) \leq d_{\ell'}(j)$, $d_{\ell''}(i) \geq d_{\ell''}(j)$, where $\{\ell', \ell''\} = \{1, 2\}$.

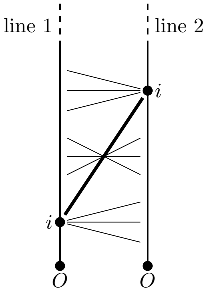
Two distinct edges are *disjoint* if and only if they do not intersect.

# Implication: non-horizontal edges

## Definition (Implication between non-horizontal edges)

Given two distinct edges $i \in N$ and $j \in N$, with $d_{\ell'}(i) < d_{\ell''}(i)$, $i$ *implies* $j$ if and only if the following three conditions are satisfied:

1. $d_{\ell'}(j) \geq d_{\ell'}(i)$,
2. $d_{\ell''}(j) \leq d_{\ell''}(i)$,
3. at least one of the two inequalities above is strict or $j < i$.



# Implication: horizontal edges

## Definition (Implication between horizontal edges)

Given two distinct edges $i \in N$ and $j \in N$, with $d_1(i) = d_2(i)$, $i$ *implies* $j$ if and only if the following three conditions are satisfied:

1. $d_1(j) = d_1(i)$,
2. $d_2(j) = d_2(i)$,
3. $j < i$.

## Property

*For any two distinct and intersecting edges $i$ and $j$, either $i$ implies $j$ or $j$ implies $i$ or both. For any two disjoint edges, none of them implies the other.*

# Primary edges

## Definition (Primary edges)

Given a line assignment,

- a non-horizontal edge is *primary* if and only if it is upward-oriented and it is not implied by any other upward-oriented edge;
- a horizontal edge is *primary* if and only if these three statements hold:
  1. it is 2-oriented,
  2. it is not implied by any upward-oriented edge,
  3. it is not implied by any 2-oriented horizontal edge.

## Property (Disjoint primary edges)

*In any line assignment all primary edges are disjoint.*

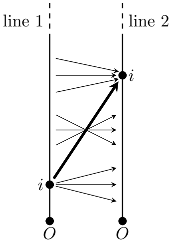
The selection of primary edges completely determines a non-dominated line assignment.

# Implications of primary edges

## Property (Non-horizontal primary edge)

*If a non-horizontal edge $i \in N$ is primary in a non-dominated line assignment $\mathcal{A}$, then*

1. *edge $i$ is upward-oriented;*
2. *each edge $j$ implying $i$ is downward-oriented;*
3. *each edge $j$ implied by $i$ is oriented to $L(i, \mathcal{A})$.*

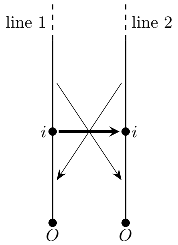


# Implications of primary edges

## Property (Horizontal primary edge)

*If a horizontal edge $i \in N$ is primary in a non-dominated line assignment $\mathcal{A}$, then*

1. *edge $i$ is $2$-oriented;*
2. *all non-horizontal edges implying $i$ are downward-oriented;*
3. *all horizontal edges implying $i$ are $1$-oriented;*
4. *all horizontal edges implied by $i$ are $2$-oriented.*



# Precedence between edges

## Definition (Partial order)

For each pair of distinct edges $i \in N$ and $j \in N$, *i precedes j* (indicated by $i \prec j$) if and only if $d_\ell(i) < d_\ell(j)\ \forall \ell = 1, 2$.

## Property

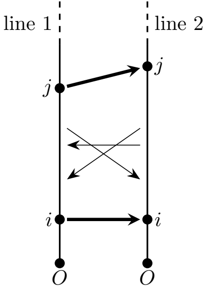
*For each pair of disjoint edges $i \in N$ and $j \in N$, either $i \prec j$ or $j \prec i$. For each pair of intersecting edges $i \in N$ and $j \in N$, neither $i \prec j$ nor $j \prec i$.*

# Implications of consecutive primary edges

## Property (Non-primary edge)

*Consider two edges $i \in N$ and $j \in N$ with $i \prec j$ that are consecutive primary edges in a non-dominated line assignment $\mathcal{A}$, i.e., there exists no primary edge $k \in N$ with $i \prec k \prec j$ in $\mathcal{A}$. Then,*

- *every non-horizontal edge $k \in N$ s.t. $i \prec k \prec j$ is downward-oriented;*
- *every horizontal edge $k \in N$ s.t. $i \prec k \prec j$ is* 1*-oriented.*



# Primary sets

As a consequence of the properties above, if the search is restricted to non-dominated line assignments, once the primary edges have been selected the orientation of all the other edges follows.

## Definition (Primary set)

The *primary set* of a solution is the set of its primary edges.

There exists a unique non-dominated line assignment $\mathcal{A}(\mathcal{P})$ having $\mathcal{P}$ as a primary set.

Primary edges are disjoint: primary sets are paths on a suitably defined digraph.

## Definition (Edge partition)

For each edge $i \in N$ we define three subsets of items in which $N$ is partitioned:

- $N^-(i) = \{j \in N : j \prec i\}$;
- $N^+(i) = \{j \in N : i \prec j\}$;
- $N^\pm(i) = N \backslash (N^-(i) \cup N^+(i))$.

## Property (Item positions)

*For any non-dominated line assignment $\mathcal{A}$ in which edge $i \in N$ is a primary edge,*

1. $d_{L(j,\mathcal{A})}(j) < d_{L(j,\mathcal{A})}(i) \;\; \forall j \in N^-(i)$;
2. $d_{L(j,\mathcal{A})}(j) > d_{L(j,\mathcal{A})}(i) \;\; \forall j \in N^+(i)$;
3. $d_{L(j,\mathcal{A})}(j) \leq d_{L(j,\mathcal{A})}(i) \;\; \forall j \in N^\pm(i)$.

# Partition of the items on each line

For any line assignment $\mathcal{A}$, let $N_\ell(\mathcal{A})$ the set of items assigned to line $\ell$:

$$N_\ell(\mathcal{A}) = \{j \in N : L(j, \mathcal{A}) = \ell\} \;\; \forall \ell = 1, 2.$$

For any given primary item $i \in N$ of $\mathcal{A}$, consider the partition of $N_\ell(\mathcal{A})$ into two subsets:

$$S_\ell(i, \mathcal{A}) = \{j \in N_\ell(\mathcal{A}) : d_\ell(j) > d_\ell(i)\}$$

and its complement

$$R_\ell(i, \mathcal{A}) = \{j \in N_\ell(\mathcal{A}) : d_\ell(j) \leq d_\ell(i)\}.$$

## Property (Independent subsets)

*If $\mathcal{A}$ is non-dominated,*

- *the elements in $S_\ell(i, \mathcal{A})$ are determined by the orientation of the edges in $N^+(i)$ and not by the orientation of the edges in $N^-(i) \cup N^\pm(i)$.*
- *the elements in $R_\ell(i, \mathcal{A})$ are determined by the orientation of the edges in $N^-(i) \cup N^\pm(i)$ and not by the orientation of the edges in $N^+(i)$.*

# Independent subsets

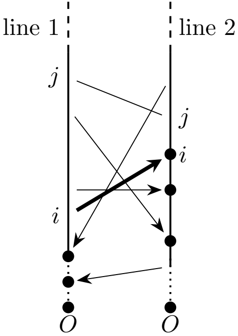



Figura: When a partial primary set is defined up to edge $i$, the orientation of all edges in $R_\ell(i, \mathcal{A})$ is defined on each line, while the orientation of all edges in $S_\ell(i, \mathcal{A})$ is unconstrained. All locations to be visited closer to $O$ than the endpoints of edge $i$ are determined (black dots), while all locations to be visited farther from $O$ than the endpoints of edge $i$ are undetermined (e.g. edge $j$).

# Independent subsets

Consider now two consecutive primary edges $i$ and $j$ in a non-dominated line assignment $\mathcal{A}$, such that $i \prec j$. Let $N_\ell^{ij}$ be the set of edges in $(N^-(j) \cup N^\pm(j)) \cap N^+(i)$ that are oriented to line $\ell$ in $\mathcal{A}$, that is,

$$N_\ell^{ij} = S_\ell(i, \mathcal{A}) \cap R_\ell(j, \mathcal{A}) \quad \forall i, j \in N \text{ primary edges s.t. } i \prec j.$$

We introduce a dummy edge 0 preceding all edges in $N$ and a dummy edge $n+1$ preceded by all edges in $N$.

We define

- $N_\ell^{0j} = R_\ell(j, \mathcal{A}) \; \forall j = 1, \ldots, n$
- $N_\ell^{i,n+1} = S_\ell(i, \mathcal{A}) \; \forall i = 1, \ldots, n$
- $N_\ell^{0,n+1} = N_\ell$.

Now $N_\ell^{ij}$ is well-defined $\forall (i, j)$ pairs with $i = 0, \ldots, n$ and $j = 1, \ldots, n+1$ and $i \prec j$.

## Property (Assignments between consecutive primary edges)

*In any non-dominated line assignment $\mathcal{A}$ in which $i \in N \cup \{0\}$ and $j \in N \cup \{n+1\}$ with $i \prec j$ are consecutive primary edges, the elements in $N_\ell^{ij}$ are determined only by the primary edges $i$ and $j$.*

# Residual items

The cost of a set of trips on a line depends on the leading items.

The leading items are determined starting from the farthest items.

The construction of partial primary sets proceeds from the origin: hence, it is not possible to determine the cost implied by the oriented edges in a partial primary set, because it is not known which items among them are leading in their trips.

However, the number of possibilities is given by $q$: enumerate them!

Given a non-dominated line assignment $\mathcal{A}$ and a primary edge $i \in N$, we define the number of *residual items* on each line at edge $i$ as the value $r_\ell(i, \mathcal{A}) \in \{0, 1, \ldots, q-1\}$ such that

$$r_\ell(i, \mathcal{A}) = |S_\ell(i, \mathcal{A})| \mod q \quad \forall \ell = 1, 2.$$

# Cost of a partial line assignment

Let $\mathcal{T}(\mathcal{A})$ be the set of trips obtained from line assignment $\mathcal{A}$.

Its cost is the sum of two contributions for each line $\ell$:

- the cost of the trips with leading item in $S_\ell(i, \mathcal{A})$;
- the cost of the trips with leading item in $R_\ell(i, \mathcal{A})$.

**Assumption:** $S_\ell(i, \mathcal{A})$ and $R_\ell(i, \mathcal{A})$ are represented as vectors indexed from 1 and sorted by non-increasing distances from $O$ on line $\ell$.

$S_\ell(i, \mathcal{A})[t]$, $R_\ell(i, \mathcal{A})[t]$: $t$-th entry of such vectors.

Sets of leading items in $S_\ell(i, \mathcal{A})$ and $R_\ell(i, \mathcal{A})$:

$$\mathcal{L}_\ell^S(i, \mathcal{A}) = \{S_\ell(i, \mathcal{A})[t] : t \mod q = 1\}$$

$$\mathcal{L}_\ell^R(i, \mathcal{A}) = \{R_\ell(i, \mathcal{A})[t] : (t + r_\ell) \mod q = 1\}$$

# Cost of a partial line assignment

$$C_\ell(S_\ell(i,\mathcal{A})) = \sum_{k \in \mathcal{L}_\ell^S} d_\ell(k).$$

This sum includes the cost terms given by the edges in $N^+(i)$ and it does not depend on the orientation of the edges in $N^-(i) \cup N^\pm(i)$.

$$C_\ell(R_\ell(i,\mathcal{A}), r_\ell) = \sum_{k \in \mathcal{L}_\ell^R} d_\ell(k).$$

This sum includes the cost terms given by the edges in $N^-(i) \cup N^\pm(i)$ and it does not depend on the orientation of the edges in $N^+(i)$, but only on the number of residual items $r_\ell(i,\mathcal{A})$ on each line.

## Cost of a partial line assignment

Setting

$$C^{+}(i, \mathcal{A}) = \sum_{\ell=1}^{2} C_\ell(S_\ell(i, \mathcal{A}))$$

$$C^{-}(i, \mathcal{A}, r_1, r_2) = \sum_{\ell=1}^{2} C_\ell(R_\ell(i, \mathcal{A}), r_\ell),$$

the cost $C(\mathcal{A})$ of solution $\mathcal{T}(\mathcal{A})$ is

$$C(\mathcal{A}) = C^{+}(i, \mathcal{A}) + C^{-}(i, \mathcal{A}, r_1, r_2).$$

# Dominance between partial line assignments

## Definition (Partial line assignments)

For any given edge $i \in N$, a *partial line assignment* $\mathcal{A}_i$ is an assignment to the lines of all items in $N^-(i) \cup N^\pm(i)$ so that $i$ is primary. The corresponding *partial primary set* $\mathcal{P}_i$ is the set of primary items of $\mathcal{A}_i$. A (partial) line assignment $\mathcal{A}$ *extends* $\mathcal{A}_i$ if and only if the items in $N^-(i) \cup N^\pm(i)$ are assigned to the same lines in both $\mathcal{A}$ and $\mathcal{A}_i$.

## Property (Dominance between partial line assignments)

*For a given $i \in N$, consider two partial line assignments $\mathcal{A}'_i$ and $\mathcal{A}''_i$ and a line assignment $\mathcal{A}''$ extending $\mathcal{A}''_i$. Let also $r_\ell = r_\ell(i, \mathcal{A}'')$ for $\ell = 1, 2$. Then if $C^-(i, \mathcal{A}'_i, r_1, r_2) < C^-(i, \mathcal{A}''_i, r_1, r_2)$ (resp. $C^-(i, \mathcal{A}'_i, r_1, r_2) = C^-(i, \mathcal{A}''_i, r_1, r_2)$), there exists $\mathcal{A}'$ extending $\mathcal{A}'_i$ such that $C(\mathcal{A}') < C(\mathcal{A}'')$ (resp. $C(\mathcal{A}') = C(\mathcal{A}'')$).*

# A dynamic programming algorithm

**Sequence.** Scan the set of items according to the parial order defined by the precedence relation $\prec$.

**State:** $\{i, r_1, r_2\}$, with

- $i \in N \cup \{0, n+1\}$
- $r_1$ and $r_2$ satisfying $(r_1 + r_2) \mod q = \rho_i$, where $\rho_i = |N^+(i)| \mod q$.

Initial states: $\{0, r_1, r_2\}$, s.t. $(r_1 + r_2) \mod q = n \mod q$.

Final state: $\{n+1, 0, 0\}$.

# A dynamic programming algorithm

**Extension rule.** Extending a state from a predecessor state $\{i, r_1, r_2\}$ to a successor state $\{j, r_1', r_2'\}$ means extending the partial line assignment $\mathcal{A}_i$ corresponding $\{i, r_1, r_2\}$ with the line assignment $\mathcal{A}_j$ such that edge $j$ is primary in $\mathcal{A}_j$ and no primary edge exists between the primary edges $i$ and $j$.

For each state $\{j, r_1', r_2'\}$, we indicate by $Pred(j, r_1', r_2')$ the set of its predecessor states:

$$Pred(j, r_1', r_2') = \{\{i, r_1, r_2\} : i \prec j \wedge r_\ell = (r_\ell' + |N_\ell^{ij}|) \mod q \ \forall \ell \in \{1, 2\}\}.$$

This defines an acyclic digraph.

# A dynamic programming algorithm

Each state $\{i, r_1, r_2\}$ has an associated cost $C(i, r_1, r_2)$, i.e. is the minimum cost of a partial solution corresponding to the state.

$C(0, r_1, r_2) = 0 \ \forall (r_1, r_2) : (r_1 + r_2) \mod q = n \mod q$.

$$C(j, r_1', r_2') = \min_{\{i, r_1, r_2\} \in Pred(j, r_1', r_2')} \{C(i, r_1, r_2) + \Delta(i, j, r_1', r_2')\}.$$

$\Delta(i, j, r_1', r_2')$: sum of the distances of the leading items in $N_1^{ij}$ and $N_2^{ij}$.

These can be identified according to the values of $r_1'$ and $r_2'$.

# A dynamic programming algorithm

For each line $\ell \in \{1, 2\}$, consider an array made of the edges in $N_\ell^{ij}$ indexed from 1 and sorted by non-increasing value of distance from $O$. Let $N_\ell^{ij}[t]$ be the edge in position $t$ in the array. Then the set of leading items in $N_\ell^{ij}$ is

$$\mathcal{L}_\ell^{ij} = \{N_\ell^{ij}[t] : (t + r'_\ell) \mod q = 1\}.$$

Then,

$$\Delta(i, j, r'_1, r'_2) = \sum_{\ell=1}^{2} \sum_{k \in \mathcal{L}_\ell^{ij}} d_\ell(k).$$

# Complexity

There are $O(nq)$ states.

The number of $(i, j)$ pairs such that $i \prec j$ is $O(n^2)$.

For each $(i, j)$ pair, $q$ values of $\Delta(i, j)$ must be computed. By a suitable procedure all of them can be computed in $O(n)$ time for each $(i, j)$ pair.

Therefore, the asymptotic worst-case time complexity of the algorithm is $O(n^3)$.